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A class of subelliptic quasilinear equations

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Abstract Suppose $\mathfrak{X} = \{X_1, X_2, \ldots, X_m\}$ is a system of real smooth vector fields on an open neighbourhood Ω of the closure of a bounded connected open set M in $\mathbb{R}^N$ satisfying the finite rank condition of Hörmander, namely the rank of the Lie algebra generated by $\mathfrak{X}$ under the usual bracket operation is a constant equal to N. We study the smoothness of solutions of a class of quasilinear equations of the form

$$Q_{\mathfrak{X}} u = \sum_{j=1}^{m} X_j^* a_j(x, u, Xu) + b(x, u, Xu) = 0$$

where $a_j, b \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^m; \mathbb{R})$. It is shown that if the matrix $\left(\frac{\partial a_j}{\partial \xi_i}\right)$ is positive definite on $M \times \mathbb{R}^{m+1}$ then any weak solution $u \in \mathcal{C}^{2,\alpha}(M, \mathfrak{X})$ belongs to $C^\infty(M)$.

Keywords Vector fields · Hörmander's finite rank condition · Quasi metric · Subelliptic quasilinear equation

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1 Preliminaries

Let M be a bounded connected open set in $\mathbb{R}^N$ and Ω be an open neighbourhood of $\overline{M}$ in $\mathbb{R}^N$. Suppose $\mathfrak{X} = \{X_1, \ldots, X_m\}$ be a system of real C^∞ vector fields on Ω. Assume that the system $\mathfrak{X}$ satisfies the finite rank condition of Hörmander (see Sect. 2). Suppose

$$a_j, b \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^m; \mathbb{R})$$

Dedicated to the memory of Sergio Companato.

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such that the $m \times m$ matrix

$$\left(\frac{\partial a_j}{\partial \xi_i}(x, u, \xi)\right)$$

is positive definite for all $(x, u, \xi) \in \Omega \times \mathbb{R}^{m+1}$. Consider the subelliptic quasilinear equation of the form

$$Q_{\mathfrak{X}}(u) = \sum_{j=1}^{m} X_j^* a_j(x, u, Xu) + b(x, u, Xu) = 0 \quad \text{in} \quad M$$

Definition 1 A weak solution of the quasilinear equation is a function $u \in L^1_{loc}(\Omega)$ such that

$$\int_M \left\{ \sum_{j=1}^{m} a_j(x, u, Xu) X_j \varphi + b(x, u, Xu)\varphi \right\} \mathrm{dx} = 0, \quad \text{for all} \quad \varphi \in \mathscr{D}(\mathrm{M})$$

1.1 Motivation

Suppose

$$Pu = \sum_{i,j=1}^{N} a_{ij}(x)\partial_{x_i}\partial_{x_j} u + \sum_{i=1}^{N} a_i(x)\partial_{x_i} u + a_0(x)u = f \quad \text{in} \quad M$$

is a linear degenerate elliptic equation in the sense that the principal symbol matrix is only positive semi definite:

$$\sum_{i,j=1}^{N} a_{ij}(x) p_i p_j \geq 0, \quad \text{for all} \quad (x, p) \in \Omega \times \mathbb{R}^N$$

with coefficients

$$a_{ij}, \quad a_i \quad a_0 \in C^\infty(\Omega) \quad \text{and} \quad a_0 < 0 \quad \text{in} \quad \Omega.$$

In any open subset ω of Ω where the rank of the principal symbol matrix $(a_{ij}(x))$ is a constant the equation $Pu = f$ can be written in the form of Hörmander's operator

$$Pu = \sum_{j=1}^{m} X_j^2 u + X_0 u + a_0(x)u = f$$

where $X_0, X_1, \ldots, X_m$ is an appropriate system of C^∞ real vector fields on Ω. The hypoellipticity of such operators under the hypothesis of finite rank was proved in the fundamental work of Hörmander [3].

1.2 Subelliptic variational problem

Quasilinear equations of the type considered here arise as the Euler-Lagrange equations in variational problems to determine stationary points of functionals of the form

$$\mathscr{F}(u) = \int_M F(x, u, Xu)\, \mathrm{dx}; \quad \text{where} \quad \mathrm{F} \in \mathrm{C}^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^{\mathrm{m}}; \mathbb{R})$$

with

$$a_j(x,u,\xi) = \frac{\partial F}{\partial \xi_j}(x,u,\xi) \quad \text{and} \quad b(x,u,\xi) = \frac{\partial F}{\partial u}(x,u,\xi)$$

2 Fundamental assumptions

For any two smooth vector fields X, Y the commutator $[X, Y]$ is the vector field defined in the standard manner

$$[X, Y] = (adX)(Y) = XY - YX$$

For $I = \{i_1, \ldots, i_k\}$ where $i_1, \ldots, i_k \in \{1, \ldots, m\}$ we shall denote by

$$X_I = [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}]]]$$

the commutator of vector fields of the system $\mathfrak{X}$ of order $|I| = k$.

(A1)—The finite rank condition of Hörmander:
There exists an integer $r \geq 1$ such that the system of vector fields $\mathfrak{X}$ together with their commutators X_I of orders $|I| = k \leq r$ span the tangent space $T_x M$ for all $x \in M$.

Consider the Lie algebra $\mathfrak{G}(\mathfrak{X})$ generated by the vector fields $\mathfrak{X} = \{X_1, \cdots, X_m\}$ under the the bracket operation $[X, Y]$, i.e. the $C^\infty(\Omega; \mathbb{R})$—module generated by the commutators X_I:

$$\mathfrak{G}(\mathfrak{X}) = \{Z = \sum_{finite} \alpha_I X_I; \alpha_I \in C^\infty(\Omega; \mathbb{R})\}$$

The condition **(A1)** is equivalent to saying that there exists an integer $r \geq 1$ such that $\forall x \in \Omega$

$$\text{rank}_x \mathfrak{G}(\mathfrak{X}) = \dim_{\mathbb{R}} \text{the vector space} \quad \{Z(x); Z \in \mathfrak{G}(\mathfrak{X})\} = N = \dim \Omega$$

Roughly speaking, the missing vector fields to form a basis of the tangent space $T_x\Omega$ at each point can be recovered by taking the commutators X_I of orders $|I|$ atmost r.

Example—In $\Omega = \mathbb{R}^2_{(x,y)}$ taking $\mathfrak{X} = \{X_1, X_2\} = \{\partial_x, x\partial_y\}$ we have $[X_1, X_2] = \partial_y$ so that $X_1, [X_1, X_2]$ generate $T_x\mathbb{R}^2$ at every point $x \in \mathbb{R}^2$ and hence $r = 2$.

We also need a technical assumption which we assume in the rest of the following:

(A2)—For each r' with $1 \leq r' \leq r$ the dimension of the space spanned by the commutators of length $|I| \leq r'$ is locally a constant.

The vector fields and their commutators can easily be expressed in local coordinates as follows: Let

$$X_i = \sum_{k=1}^{N} h_i^k(x)\partial_{x_k}, \quad i = 1, \ldots, m$$

Then the commutator is given by

$$[X_i, X_j] = \sum_{k,l=1}^{N} \{h_i^k(x)(\partial_{x_k} h_j^l(x))\partial_{x_l} - h_j^l(x)(\partial_{x_l} h_i^k(x))\partial_{x_k}\}$$

The distribution adjoint of X_j can be calculated as follows:

$$\left\langle X_j^* u, \varphi \right\rangle = \left\langle u, X_j \varphi \right\rangle = \left\langle u, \sum_{k=1}^{N} h_j^k(x) \partial_{x_k} \varphi \right\rangle$$

$$= -\left\langle \sum_{k=1}^{N} \partial_{x_k}(h_j^k(x)u), \varphi \right\rangle = -\left\langle X_j u, \varphi \right\rangle - \left\langle \left(\sum_{k=1}^{N} \partial_{x_k} h_j^k \right) u, \varphi \right\rangle$$

and hence

$$X_j^* = -X_j - \tilde{h}_j(x), \quad \text{with} \quad \tilde{h}_j \in C^\infty(\Omega).$$

3 The geometry associated to the system of vector fields $\mathfrak{X}$

Following E.M. Stein and L.P. Rothschield, the assumptions **(A1), (A2)** lead to introduce a quasi-metric $\rho(x, y)$ defined locally on Ω as follows;

For each $x \in \Omega$ we can choose a family of vector fields

$$\{X_{jk};\, 1 \le j \le r, 1 \le k \le k_j\}$$

where $\{X_{jk}\}$ is a commutator of order $= j$, $1 \le k \le k_j$ such that for any $1 \le r' \le r$ the space spanned by the family $\{X_{jk};\, j \le r',\ 1 \le k \le k_j\}$ is precisely the space spanned by all the commutators of orders $\le r'$. We may assume $\{X_{jk}\}$ to be linearly independent. If $E_{r'}$, for $r' \le r$, denotes the vector space spanned by all commutators of orders $\le r'$ then

$$E_1 \subset E_2 \subset \ldots \subset E_r = T_x(\Omega)$$

.

By the local existence and uniqueness theorem for ordinary differential equations it follows that, for each $x \in \Omega$ there exists a neighbourhood V_x (where the exponential map is well defined) such that any $y \in V_x$ can be represented as

$$y = \exp\left(\sum a_{jk} X_{jk} \right)(x)$$

The mapping $\Theta\colon y \to (a_{jk})$ is a local C^∞-diffeomorphism of V_x to a neighbourhood of the origin in $\mathbb{R}^N$.

$$(a_{jk});\, 1 \le j \le r;\, 1 \le k \le k_j$$

are called canonical coordinates of y with respect to the choice of the system (X_{jk})

Remark 1 In the Euclidean space $\mathbb{R}^N$, taking $\mathfrak{X} = \{X_1, \ldots, X_N\} = \left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N} \right\}$ we have $r = 1$, $\rho(x, y) = d_E(x, y) = \sqrt{\sum (y_j - x_j)^2}$ and $y_j - x_j = \exp(a_j \frac{\partial}{\partial x_j}(x))$, i.e. $y = x + \sum a_j \mathrm{e_j}$ and $(a_1, \ldots, a_N)$ are the canonical coordinates of y of origin x with respect to the system $\mathfrak{X}$.

Definition 2 (*E.M. Stein and L.P. Rothschield*) The local quasi-metric ρ on Ω is defined for $y \in V_x$ by

$$\rho(x, y) = \left(\sum |a_{jk}|^{\frac{2r!}{j}} \right)^{\frac{1}{2r!}}$$

We denote, for $R > 0$ the ρ-ball of centre x and radius R by

$$B_\rho(x, R) = \{y \in V_x;\ \rho(x, y) < R\}$$

3.1 Properties of the local quasi-metric ρ

The local quasi-metric $\rho(x, y)$ has many properties similar to that of the Euclidean metric, some of these useful properties are recalled below:

Let V_{x_0} be the local diffeomorphism neighbourhood of a point $x_0 \in \Omega$ used to define the local metric ρ.

(a) There exists a constant $C > 0$ such that

$$C^{-1}|x - y| \le \rho(x, y) \le C|x - y|^{\frac{1}{r}}, \quad \forall x, y \in V_{x_0}$$

where $r \ge 1$ is the order of the commutators in the condition of Hörmander (Assumption **(A1)**)

For $\delta_0 > 0$ such that $B_\rho(x, \delta_0) \subset V_{x_0}$ we have

$$\text{the Lebesgue measure of } \ B_\rho(x, \delta) = |B_\rho(x, \delta)| \le \text{const.}|\lambda(x)|\delta^{\mathrm{d}}, \quad \text{for} \quad 0 < \delta \le \delta_0$$

where

$$\lambda(x) = \det(X_{jk}) \quad \text{and} \quad \mathrm{d} = \sum_{j \le r} j k_j$$

We also recall that $N = \sum_{j \le r} k_j$.

Moreover any compact set $K \subset V_{x_0}$ can be covered by a finite number of ρ-balls $B_\rho(x, \delta_0)$.

For any compact set $K \subset V_{x_0}$ there exists an $\epsilon_0 > 0$ such that

$$B_E(x, c_1\epsilon^r) \subset B_\rho(x, \epsilon) \subset B_E(x, c_2\epsilon). \quad 0 < \epsilon \le \epsilon_0$$

where B_E stands for the Euclidean ball, with some constants c_1 and c_2 [5,7].

(b) *Regularity of* ρ: $\rho \in C^\infty(V_{x_0} \times V_{x_0})$ and for any $J = (j_1, \ldots, j_k)$ with $j_1, \ldots, j_k \in \{1, 2, \ldots, m\}$ we have

$$|X^J \rho(x, y)| = |X_{j_1} X_{j_2} \ldots X_{j_k} \rho(x, y)| \le c_J \rho(x, y)^{1-|J|}$$

Here the derivations $X'_{j_i}s$ act either in the variables x or in the variables y in V_{x_0}.

(c) For any compact set $K \subset V_{x_0}$ such that

$$K_{3\epsilon} = \{x \in \mathbb{R}^N;\ \ \rho(x, y) < 3\epsilon,\ y \in K\} \subset V_{x_0}$$

there exists a smooth test function $\psi \in \mathscr{D}(K_{3\epsilon})$ such that

$$0 \le \psi(x) \le 1, \quad \text{and} \quad \psi(x) = 1 \quad \text{on} \quad K,$$

and we have

$$|X^J \psi(x)| \le C_J \epsilon^{-|J|} \quad \text{for all} \quad J = \{j_1, \ldots, j_k\}$$

(d) *Cut off and regularizing functions in the geometry asociated to the system* $\mathfrak{X}$:

Let

$$h(t) = \begin{cases} \exp\left(\frac{1}{t^2-1}\right) & \text{for } \ t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and define

$$\varphi(x) = \epsilon^{-N} h\left(\frac{\rho(x,0)}{\epsilon}\right) \in \mathscr{D}(\rho(x,0) \leq \epsilon)$$

4 Function spaces associated to the system of vector fields

In order to study weak solutions of the quasilinear equation $Q_{\mathfrak{X}} u = 0$ we introduce the following natural spaces of Sobolev type:

Let $p \geq 1$ and $k \in \mathbb{N}$. Define

$$W^{k,p}(M, \mathfrak{X}) = \{f \in L^p(M); X^J f \in L^p(M), \quad \text{for all} \quad J = \{j_1, \ldots, j_k\}\}$$

where $X^J f = X_{j_1} X_{j_2} \cdots X_{j_k} f$ with its natural norm

$$||f, W^{k,p}(M, \mathfrak{X}|| = \left(\sum_{|J| \leq k} ||X^J f, L^p(M)||^p\right)^{\frac{1}{p}}$$

$W^{k,p}(M, \mathfrak{X})$ is a Banach space which is reflexive if $1 < p < +\infty$ and is seperable for $1 \leq p < +\infty$.

$W^k(M, \mathfrak{X}) = W^{k,2}(M, \mathfrak{X})$ is Hilbert space

and

$W_0^{k,p}(M, \mathfrak{X})$ is the closure of the space of smooth functions $\mathscr{D}(M)$ in $W^{k,p}(M, \mathfrak{X})$.

The dual space of $W_0^{k,p}(M, \mathfrak{X})$ is the space of all distributions on M which can be represented (in a non unique way) as a distribution

$$g_0 + \sum_{j=1}^{m} X_j^* g_j \quad \text{with} \quad g_0, g_1, \ldots, g_m \in L^{p'}(M), \quad \frac{1}{p} + \frac{1}{p'} = 1$$

where $X_j^* g_j$ is taken in the sense of distributions , that is

$$f \to \int_M \left(g_0 f + \sum_{j=1}^{m} g_j X_j f\right) \mathrm{dx}, \quad \text{for all} \quad \mathrm{f \in W_0^{k,p}(M, \mathfrak{X})}$$

4.1 Properties

Assume that $\mathfrak{X}$ satisfies the assumptions (A1) and (A2).

(a) For all $k \geq 1$ and $p \geq 1$ we have the continuous inclusion $W_0^{k,p}(M, \mathfrak{X}) \to W^{\frac{k}{r},p}(M)$, the classical Sobolev space.

This is a consequence of hypoellipticity of the subelliptic operator of Hörmander

$$\mathcal{L}_{\mathfrak{X}} = \sum_{j=1}^{m} X_j^* X_j + c(x), \quad \text{with} \quad c \in C^\infty(M)$$

(b) Using the classical embedding theorem for Sobolev spaces we get the embedding

$$W_0^{k,p}(M, \mathfrak{X}) \to L^{\overline{p}}(M) \quad \text{where} \quad \frac{1}{\overline{p}} = \frac{1}{p} - \frac{k}{dr} \quad \text{when} \quad kp < dr$$

and

$$W_0^{k,p}(M,\mathfrak{X}) \to C^l(M) \quad \text{when} \quad \frac{k}{r} - \frac{d}{p} > l \geq 0$$

with a constant of inclusion $= const.[meas.(\Omega)]^{\frac{k}{r}-\frac{d}{p}}$.

(a$'$) There exists an $s > 0$ (independent of $p \geq 1$) such that for any compact subset K in M there is a constant $c = c(p,k)$ so that

$$\|u, W^{s,p}(M)\| \leq c\|u, W^{1,p}(M,\mathfrak{X})\| \quad \text{for any} \quad u \in \mathscr{D}(K)$$

(c) *Interpolation lemma*—If M is a subdomain of Ω with C^∞ - boundary then for any $\epsilon > 0$ and $0 < |J| < k$ there exists a constant $c(\epsilon,k)$ such that

$$\|X^J u, L^p(M)\| \leq \epsilon\|u, W^{k,p}(M,\mathfrak{X})\| + c(\epsilon,k)\|u, L^p(M)\|$$

(d) *Poincaré inequality*—

(i) For any $x_0 \in M$ there exists an R_0 such that

$$\|\varphi, L^p(B_\rho(x_0,R))\| \leq cR\sum_{j=1}^{m}\|X_j\varphi, L^p(B_\rho(x_0,R))\|$$

for all $\varphi \in W_0^{1,p}(B_\rho(x_0,R),\mathfrak{X})$, where $0 < R \leq R_0$

(ii) Suppose there is atleast one vector field X_j in the system $\mathfrak{X}$ which can be globally straightenned in M then

$$\|\varphi, L^p(M)\| \leq c(diam M)\sum_{j=1}^{m}\|X_J\varphi, L^p(M)\| \quad \text{for all} \quad \varphi \in W_0^{1,p}(M,\mathfrak{X})$$

(iii) (L.P. Rothschield and E.M. Stein)—There exist $c > 0$ and $R_0 > 0$ such that, for all $x_0 \in M$ with $B_\rho(x_0,2R) \subset V_{x_0}$, we have

$$\int_{B_\rho(x_0,R)} |u(x) - u_R|^p \,\mathrm{dx} \leq \mathrm{cR^p}\int_{\mathrm{B_\rho(x_0,R)}}\sum_{\mathrm{j=1}}^{\mathrm{m}}|\mathrm{X_j u}|^{\mathrm{p}}\,\mathrm{dx}$$

where $u_R = \frac{1}{|B_\rho(x_0,R)|}\int_{B_\rho(x_0,R)} u(y)\,\mathrm{dy}$, and $|B_\rho(x_0,R)|$ denotes the Lebesgue measure of the ρ ball $B_\rho(x_0,R)$.

(e) *Truncation method of Stampacchia*—

In the proofs of boundedness and that of Hölder continuity of weak solutions of the quasilinear subelliptic equation an essential idea is the use of the technique of Stampacchia, namely that of using suitable truncations of the solution itself as test functions in the definition of weak solutions [4,8]. We have the following result due to C.-J. Xu:

Theorem 1 *(C.-J. Xu) Let $u \in W_0^{1,p}(M,\mathfrak{X})$ and $k \geq 0$. Then*

$$u^{(k)}(x) = \max(u(x) - k, 0)$$

belongs to $W_0^{1,p}(M,\mathfrak{X})$ and more over

$$X_j u^{(k)}(x) = \begin{cases} X_j u(x) & a.e. \ in \ \ E_k = \{x \in M; u(x) > k\} \\ 0 & elsewhere \end{cases}$$

The proof consists in first proving the result for an approximating sequence of smooth functions $\{u_\nu\}$ in $\mathscr{D}(M)$. Since $\{X_j u_\nu^{(k)}\}$ is a bounded set in $L^p(M)$ one uses the standard weak convergence arguments combined with some delicate measure theoretical arguments as in Stampacchia.

In order to prove C^∞-regularity of solutions of linear and then of qusilinear equations of subelliptic type we shall use function spaces of Hölder type adapted to the geometry defined by the system of vector fields.

5 Function spaces of Hölder type in the geometry associated to the system of vector fields $\mathfrak{X}$

We denote by

$$\mathscr{C}(V, \mathfrak{X}) = C^0(\overline{V})$$

For $\alpha \in (0, 1)$

$$\mathscr{C}^{0,\alpha}(V, \mathfrak{X}) = \{f \in C^0(\overline{V});\quad |f(x) - f(y)| \le c\rho(x, y)^\alpha\}$$

and is provided with the semi-norm

$$[f]_{\mathfrak{X},\alpha} = [f]_{\mathfrak{X},\alpha,V} = \sup_{x,y\in V, x\neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\alpha}$$

For $k \in \mathbb{N}$ and $\alpha \in [0, 1]$,

$$\mathscr{C}^{k,\alpha}(V, \mathfrak{X}) = \{f \in \mathscr{C}^{0,\alpha}(V, \mathfrak{X});\, X^J f = X_{j_1} \dots X_{j_h} f \in \mathscr{C}^{0,\alpha}(V, \mathfrak{X}),$$

$$\text{for all}\quad J = (j_1, \dots, j_h),\quad \text{with}\quad |j| = h \le k\}$$

let

$$[f]_{\mathfrak{X},k,0} = \sum_{|J|=k} \sup_{x\in V} |X^J f(x)|$$

$$[f]_{\mathfrak{X},k,\alpha} = \sum_{|J|=k} [X^J f]_{\mathfrak{X},\alpha}$$

We define the norm

$$\|f, \mathscr{C}^{k,\alpha}(V, \mathfrak{X})\| = \sum_{j=0}^{k} [f]_{\mathfrak{X},j,0} + [f]_{\mathfrak{X},k,\alpha}$$

The properties of $\rho(x, y)$ give the following standard properties of these function spaces. Recall that $r \ge 1$ is the number such that the commutators of length $\le r$ span the tangent space at each point.

1. $\mathscr{C}^{k,\alpha}(V, \mathfrak{X})$ is a Banach space.
2. $\mathscr{C}^{0,\alpha}(V, \mathfrak{X})$ is continuously embedded in $C^{0,\frac{\alpha}{r}}(V)$, the classical space of Hölder continuous functions of exponent $\frac{\alpha}{r}$.
3. $\mathscr{C}^{kr,0}(V, \mathfrak{X})$ is continuously embedded in $C^{k,1}(V)$, the space of all $C^k(V)$ functions with all the k-th order derivatives Lipschitz continuous in V.

4. If F: $V\times\mathbb{R}^m \to \mathbb{R}$ is a C^∞-function and $z \in \mathscr{C}^{k,\alpha}(V,\mathfrak{X})$ then $F(x,z(x)) \in \mathscr{C}^{k,\alpha}(V,\mathfrak{X})$.
5. If $(k,\alpha), (j,\beta) \in \mathbb{N}\times[0,1]$ are such that $k+\alpha > 0$ and $j+\beta < k+\alpha$ then any bounded set $\mathscr{B}$ in $\mathscr{C}^{k,\alpha}(V,\mathfrak{X})$ is relatively compact in $\mathscr{C}^{j,\beta}(V,\mathfrak{X})$.
6. Interpolation inequality - Suppose $(k,\alpha), (j,\beta) \in \mathbb{N}\times[0,1]$ are such that $k+\alpha > 0$ and $j+\beta < k+\alpha$. Then for any $\epsilon > 0$ there exists a constant $c_\epsilon = c(\epsilon, j, k, V, r)$ such that

$$\|u, \mathscr{C}^{j,\beta}(V,\mathfrak{X})\| \le \epsilon\|u, \mathscr{C}^{k,\alpha}(V,\mathfrak{X})\| + C_\epsilon\|u, L^\infty(V)\|$$

6 Minimization problem for the functional $\mathscr{F}(u)$

Suppose $F : \Omega\times\mathbb{R}\times\mathbb{R}^m \to \mathbb{R} \in C^0$ and satisfies the following conditions:

(i) $\xi \to F(x,u,\xi)$ is a convex function for all $(x,u) \in \overline{M}\times\mathbb{R}$;
(ii) $F(x,u,\xi)$, $\frac{\partial F}{\partial u}(x,u,\xi)$ and $\frac{\partial F}{\partial \xi_j}(x,u,\xi)$ are continuous functions on $\overline{M}\times\mathbb{R}\times\mathbb{R}^m$;
(iii) there exist $p > 1$ and a constant $c_0 > 0$ such that

$$F(x,u,\xi) \ge c_0\|\xi, \mathbb{R}^m\|^p.$$

Then we have the following existence theorem:

Theorem 2 *Assume that the system of vector fields $\mathfrak{X}$ satisfies the assumptions (A1) and (A2), and further that there is atleast one vector field of the system $\mathfrak{X}$ which can be straightenned globally in M. If F satisfies the conditions (i), (ii) and (iii) and if there exists a function $v \in W_0^{1,p}(M,\mathfrak{X})$ such that $\mathscr{F}(u) < +\infty$ then there exists a minimum $u \in W_0^{1,p}(M,\mathfrak{X})$ for the functional $\mathscr{F}$.*

In order to prove the regularity of the minimizing solution of the subelliptic variational problem it is natural to seek conditions on F defining the functional $\mathscr{F}$ so that the solution of the variational problem is also the weak solution of the associated Euler–Lagrange equation, which is of the form of the subelliptic equation $Q_{\mathfrak{X}}(u) = 0$ with

$$a_j(x,u,\xi) = \frac{\partial F}{\partial \xi_j}(x,u,\xi) \quad \text{and} \quad b(x,u,\xi) = \frac{\partial F}{\partial u}(x,u,\xi)$$

For simplicity we consider the case $p = 2$.

Theorem 3 *Suppose the function F satisfies the following conditions:*
There exist functions $h_0 \in L^1(M)$, $h_1 \in L^{\overline{2}}(M)$ and $h_2 \in L^2(M)$ where

$$\frac{1}{\overline{2}} = \frac{1}{2} - \frac{1}{Nr} \quad \textit{and} \quad \frac{1}{\overline{2}} + \frac{1}{(\overline{2})'} = 1$$

such that

$$|F(x,u,\xi)| \le const.\{\|\xi, \mathbb{R}^m\|^2 + |u|^{\overline{2}} + h_0(x)\}$$

$$|\frac{\partial F}{\partial u}(x,u,\xi)| \le const.\{\|\xi, \mathbb{R}^m\|^{\frac{2}{(\overline{2})'}} + |u|^{\overline{2}-1} + h_1(x)\}$$

$$|\frac{\partial F}{\partial \xi_j}(x,u,\xi)| \le const.\{\|\xi, \mathbb{R}^m\| + |u|^{\frac{q}{2}} + h_2(x)\}$$

with some $1 < q < \overline{2}$. *Then the minimizer of the functional* $\mathscr{F}$ *is also the weak solution of the quasilinear equation*

$$\sum_{j=1}^{m} X_j^* \frac{\partial F}{\partial \xi_j}(x, u, Xu) + \frac{\partial F}{\partial u}(x, u, Xu) = 0$$

.

The proof consists of a straight forward computation of the first variation of the functional and the assumptions on the function F justify the necessary derivations.

7 Boundedness of the minimizer of the functional $\mathscr{F}(u)$

The first step in the study of regularity of the minimizer is to prove the boundedness of the solution u. This requires growth conditions on the function F. More precisely we have the following result:

Theorem 4 *Suppose* $\mathfrak{X}$ *satisfies the assumption (A1), namely the finite rank condition of Hörmander and that there is atleast one vector field of the system* $\mathfrak{X}$ *which can be globally straightenned in M. Suppose given* $f \in W^1(M, \mathfrak{X}) \cap L^\infty(\partial M)$ *and let* $\|u, L^\infty(\partial M)\| \leq M_0$. *Assume that the function F satisfies the following growth conditions: there exist constants* $c_1 \geq 0$ *and* $c_2 \geq 0$ *such that*

$$\begin{cases} F(x, u, \xi) \geq c_1 \|\xi, \mathbb{R}^m\|^2 - c_2|u|^q - f(x)|u|^2 \\ F(x, u, 0) \geq c_2|u|^q + f(x)|u|^2 \end{cases}$$

for $|u| \geq M_0$ *where* $2 < q < \overline{2} = \frac{2Nr}{Nr-2}$. *If* $u \in W^1(M, \mathfrak{X})$ *is the minimizer of the functional* $\mathscr{F}$ *on the affine subspace* $\{v \in W^1(M, \mathfrak{X}); \quad v - f \in W_0^1(M, \mathfrak{X})\}$ *then* $u \in L^\infty(M)$ *and is bounded by a constant depending on* $N, r, \|f\|_{L^2}, \|Xf\|_{L^2}, meas M, c_1, c_2$.

The proof makes use of the truncations $u^{(k)}$ with $k \geq M_0$ and the fact that $u^k = u - u^{(k)}$ belongs to the affine subspace and comparing the functional and estimating using the growth conditions [9,10].

8 Hölder continuity of weak solutions of the quasilinear equation $Q_{\mathfrak{X}}(u) = 0$

Let M be a bounded connected open subset of Ω. Assume that $\mathfrak{X}$ satisfies the assumptions (A1) and (A2). Suppose $a_j, b \in C^\infty(\Omega \times \mathbb{R}^{m+1})$ and satisfy the following structure conditions: there exist positive continuous functions $f, g \colon \Omega \to \mathbb{R}_+$ such that

$$\begin{cases} \sum_{j=1}^{m} a_j(x, u, \xi)\xi_j \geq c_0\|\xi, \mathbb{R}^m\|^2 - g(x)^2 \\ |a_j(x, u, \xi)| \leq c_1\{\|\xi, \mathbb{R}^m\| + g(x)\} \\ |b(x, u, \xi)| \leq c_2\{\|\xi, \mathbb{R}^m\|^2 + f(x)\} \\ \text{for} \quad (x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^m \end{cases}$$

Theorem 5 *(C.-J. Xu) If* $u \in W^1(M, \mathfrak{X})$ *is a weak solution of the quasilinear equation*

$$Q_{\mathfrak{X}}(u) = \sum_{j=1}^{m} X_j^* a_j(x, u, Xu) + b(x, u, Xu) = 0$$

i.e. $u \in W^1(M, \mathfrak{X})$ *and for all* $\varphi \in \mathscr{D}(M)$,

$$\int_M \left\{ \sum_{j=1}^{m} a_j(x, u, Xu)(X_j\varphi) + b(x, u, Xu)\varphi \right\} \mathrm{dx} = 0$$

then for any connected sub-domain $\omega \subset\subset M$ *there exist constants* $c > 0$ *and* $0 < \lambda < 1$ *such that*

$$\|u, C^{0,\lambda}(\omega)\| \leq c \left\{ \sup_{x \in M} |u(x)| + K(f, g) \right\}$$

with $K(f, g)$ *depending on the norms of* $f + g^2$ *in* $L^{\tilde{q}}(M)$ *and of* g *in* $L^t(M)$ *where*

$$\frac{1}{\tilde{q}} = \frac{2}{q} - \frac{1}{N} + \frac{r}{q} \quad \text{and} \quad \frac{1}{t} = \frac{1}{q} + \frac{r}{q} - \frac{1}{N}$$

for some $q \in (Nr, N(r+1))$.

8.1 A brief idea of the main steps in the proof

The proof is rather long but uses the standard arguments as in the theory of second order elliptic equations due to Stampacchia and Moser. First of all the weak solution is shown to be bounded using its truncations $u^{(k)}$ as test function in the definition of the weak solution. Next by the method of Moser one proves a Harnack type inequality from which one obtains the Hölder continuity of the weak solution.

9 C^∞-regularity for subelliptic quasilinear equations

Here we are concerned with the interior C^∞ regularity of solutions of the subelliptic quasi-linear equation

$$\mathscr{E}_{\mathfrak{X}}(u) = \sum_{i,j=1}^{m} a_{ij}(x, u, Xu)X_iX_ju + b(x, u, Xu) = 0$$

where $a_{ij}, b \in C^\infty(\Omega \times \mathbb{R}^{m+1})$ and for $(x, u, \xi) \in \Omega \times \mathbb{R}^{m+1}$, the $(m \times m)$ matrix $a_{ij}(x, u, \xi)$ is positive definite and we assume the fundamental hypothesis (A1) and (A2) on the system $\mathfrak{X}$. As we are interested in the local behaviour of solutions we may work in a neighbourhood V_{x^0} of a point $x^0 \in M$ where the exponential map (a local diffeomorphism) used to define the quasi-metric $\rho(x, y)$ is well defined.

We make use of the function spaces of Hölder type adapted to the geometry defined by the system of vector fields which satisfy the fundamental assumptions (A1) and (A2) (see Sect. 4). The first step in the proof of regularity consists in obtaining Schauder type estimates in these spaces for solutions of the linear equation associated to Hörmander's operator $\mathscr{L}_{\mathfrak{X}}$, which in turn follow from the existence and estimates for the associated Green's kernel.

9.1 The interior regularity of solutions

We are now in a position to formulate the main interior regularity result for solutions of the second order subelliptic quasilinear equation of the form $\mathscr{E}_{\mathfrak{X}}(u) = 0$.

Theorem 6 *Assume that the system of vector fields satisfy the assumptions (A1) and (A2). If* $u \in \mathscr{C}^{2,\alpha}(V, \mathfrak{X})$ *is a solution of the quasilinear equation*
$\mathscr{E}_{\mathfrak{X}}(u) = 0$ *where*

$$a_{ij}, b \in C^{\infty}(V \times \mathbb{R} \times \mathbb{R}^m)$$

and

$$(a_{ij}(x, u, \xi)) \text{ is an } m \times m \text{ positive definite matrix for all } (x, u, \xi) \in V \times \mathbb{R} \times \mathbb{R}^m$$

then $u \in C^{\infty}(V)$.

10 Sketch of proof of the main interior regularity

The proof is an adaptation of the proof of the classical regularity theorem for linear and quasilinear elliptic equations based on the use of the Green's operator (rather the Green's kernel) and the Schauder type estimates derived from estimates on the Green's kernel.

(a) **Hörmander's operator $\mathscr{L}_{\mathfrak{X}}$ and its Green's operator**

Assume the system of vector fields $\mathfrak{X}$ satisfies the hypotesis (A1) and (A2) and consider the operator of Hörmander

$$\mathscr{L}_{\mathfrak{X}} = \sum_{j=1}^{m} X_j^2 + c(x), \quad \text{with } c \in C^{\infty}(\Omega), \quad c(x) \leq c_0 < 0$$

Definition 3 A positive operator $G\colon C^0(\overline{M}) \to C^0(\overline{M})$ is said to be the Green's operator for $\mathscr{L}_{\mathfrak{X}}$ with the Dirichlet boundary condition if

$$f \to u = Gf \text{ is a solution of the equation } \mathscr{L}_{\mathfrak{X}} u = -f \text{ in } M$$

and satisfies the homogeneous Dirichlet condition $u = 0$ on ∂M.

By the hypoellpticity theorem of Hörmander for $\mathscr{L}_{\mathfrak{X}}$, under the hypothesis (A1) we know that if, in an open set $\omega \subset M$, $f \in C^{\infty}(\omega)$ then $u = Gf \in C^{\infty}(\omega)$.

Next, by the kernel theorem of Schwartz, G has a distribution kernel $g \in \mathscr{D}'(M \times M)$, called the Green's kernel for $\mathscr{L}_{\mathfrak{X}}$:

$$\langle g, \varphi \otimes f \rangle = \langle Gf, \varphi \rangle, \quad \text{for all } f, \varphi \in \mathscr{D}(M)$$

We have the following result due to Bony [1] and [2]:

Theorem 7 *There exists a Green's kernel* $g \in \mathscr{D}'(M \times M)$ *for* $\mathscr{L}_{\mathfrak{X}}$ *such that*

1. $g(x, y) > 0$ *in the sense of distributions on* $M \times M$;
2. $g \in C^{\infty}((M \times M) \backslash diagonalin\ \ M))$;
3. $f \to Gf(x) = \int_{\omega} g(x, y) f(y)\,\mathrm{d}y$, *for* $f \in C^{\infty}(\overline{\omega})$ *for* $\omega \subset\subset M$ *is the Green's operator for* $\mathscr{L}_{\mathfrak{X}}$;

Moreover, when $N \geq 3$, *we have*

(a)

$$|X^J g(x, y)| \leq c_J |B_{\rho}(x, \rho(x, y))|^{-1} \rho(x, y)^{2-|J|},$$

where

$$J = (j_1, \dots, j_l), \quad j_i \in \{1, \dots, m\}$$

where the differentiations X_{j_i} act either in the variables x or in the variables y and

(b) *in a neighbourhood of the diagonal Δ_M in $M \times M$*

$$-g(x, y) \geq c' \rho(x, y)^2 |B_\rho(x, \rho(x, y))|^{-1}$$

We assume from now on that M is itself the exponential mapping neighbouhood $V = V_{x^0}$ of a point $x^0 \in M$.

Theorem 8 *Let $R > 0$ be such that $B_\rho(x^0, 2R) \subset V$. If $f \in \mathscr{C}^{0,\alpha}(V, \mathfrak{X})$ with supp.$f \subset B_\rho(x^0, R)$ with $\alpha > 0$ define*

$$v(x) = Gf(x) = \int_{B_\rho(x^0, R)} g(x, y) f(y)\, \mathrm{dy}$$

Then $f \to Gf$ is a continuous linear mapping of $\mathscr{C}^{0,\alpha}(B_\rho(x, R), \mathfrak{X})$ into $\mathscr{C}^{2,\alpha}(B_\rho(x, R), \mathfrak{X})$

Proof Using the estimate (a) for g we can differentiate under the integral sign to get

$$X_j v(x) = \int_{B_\rho(x^0, R)} (X_j^x g)(x, y) f(y)\, \mathrm{dy}$$

(Here $X_j^x g$ means that the vector field X_j acts as a derivation in x). To estimate the second derivatives $X_i X_j v$ it is standard to avoid a neighbourhood of the diagonal using a cut off function. Let

$\eta \in C^\infty(\mathbb{R})$ be such that $0 \leq \eta \leq 1$ and $\eta(t) = 0$ in $t \leq 1$, $\eta(t) = 1$ in $t \geq 2$. Define

$$\eta_\epsilon(x, y) = \eta\left(\frac{\rho(x, y)}{\epsilon}\right) = 0 \;\text{ when }\; \rho(x, y) \leq \epsilon \;\text{ and }\; \eta_\epsilon(x, y) = 1 \;\text{ when }\; \rho(x, y) \geq 2\epsilon$$

and

$$v_\epsilon(x) = \int_{B_\rho(x^0, R)} \eta_\epsilon(x, y) g(x, y) f(y)\, \mathrm{dy} \in \mathrm{C}^1(\mathrm{B}_\rho(\mathrm{x}^0, \mathrm{R}))$$

Then it is easy to check, using the estimate (a) for g, that

$$v_\epsilon \to v \;\text{ and }\; X_j v_\epsilon \to X_j v \;\text{ as }\; \epsilon \to 0, \quad j = 1, \dots, m$$

In fact we have

$$\int_{B_\rho(x^0, R)} |(X_j v_\epsilon)(x) - (X_j v)(x)|\, \mathrm{dx} \leq \mathrm{c}\epsilon \sup_{\mathrm{B}_\rho(\mathrm{x}^0, \mathrm{R})} |\mathrm{f}(\mathrm{x})| = \mathrm{c}\epsilon \|\mathrm{f}; \mathscr{C}^0(\mathrm{B}_\rho(\mathrm{x}^0, \mathrm{R}), \mathfrak{X})\|$$

A somewhat *technical estimate* by making use of a result of Rothschield and Stein [6] one obtains

$$(X_i X_j v(x) = \int_{B_\rho(x^0, R)} |(X_i^x X_j^x g)(x, y)[f(x) - f(y)]\, \mathrm{dy} + \mathrm{f}(\mathrm{x}) \int_{\mathrm{B}_\rho(\mathrm{x}^0, \mathrm{R})} \mathrm{g}_{\mathrm{ij}}^0(\mathrm{x}, \mathrm{y})\, \mathrm{dy}$$

where g_{ij}^0 satisfies an estimate similar to that for $X^J g$ with $|J| = 2$ and from this we obtain the following estimate:

$$|(X_i X_j v)(x) - (X_i X_j v)(x')| \leq c \rho(x, x')^\alpha . [f]_{\mathfrak{X}, \alpha, B_\rho(x^0, R)}.$$

By a bootstrap argument we then get the following □

Theorem 9 *If* $f \in \mathscr{C}^{k,\alpha}(V, \mathfrak{X})$ *with* $k \in \mathbb{N}$ *and* $\alpha \in (0, 1)$ *with* $supp.f \subset B_\rho(x^0, R)$ *then* $v \in \mathscr{C}^{k+2,\alpha}(V, \mathfrak{X})$ *and we have the estimate*

$$\|v, \mathscr{C}^{k+2,\alpha}(V, \mathfrak{X})\| \leq const.\|f; \mathscr{C}^{k,\alpha}(V, \mathfrak{X})\|$$

This result extends to *inhomogeneous equation* of Hörmander:

$$\mathscr{L}_{\mathfrak{X}} u = \sum_{j=1}^{m} X_j^2 u + c(x)u = f, \quad \text{with} \quad f, c \in C^\infty \quad \text{and} \quad c(x) \leq c_0 < 0$$

by writing $u = v + w$ where

$$\mathscr{L}_{\mathfrak{X}} v = 0 \quad \text{and} \quad w(x) = \int g(x, y) f(y)\, \mathrm{d}y.$$

For the equation $\mathscr{L}_{\mathfrak{X}} u = f$ in $V = V_{x^0}$, take $t, s \in \mathbb{R}$ with $0 < t < s \leq 1$ and

$$\zeta \in \mathscr{D}(B_\rho(x^0, sR)) \quad \text{such that} \quad \zeta(x) = 1 \quad \text{in} \quad B_\rho(x^0, tR).$$

Then

$$\mathscr{L}_{\mathfrak{X}}(\zeta u) = \zeta(\mathscr{L}_{\mathfrak{X}} u) - \sum_{j=1}^{m} [(X_j^* \zeta)(X_j u) + (X_j \zeta)(X_j^* u)] - (\mathscr{L}_{\mathfrak{X}}(\zeta))u$$

and we conclude to obtain

Theorem 10 *If* $f \in \mathscr{C}^{k,\alpha}(V, \mathfrak{X})$ *for some* $k \in \mathbb{N}$ *and* $\alpha \in (0, 1)$ *and if* $u \in \mathscr{C}^{2,\alpha}(V, \mathfrak{X}))$ *is a weak solution of* $\mathscr{L}_{\mathfrak{X}} u = f$ *then* $u \in \mathscr{C}^{k+2,\alpha}(V, \mathfrak{X})$ *and satisfies an estimate of the form*

$$\|u, \mathscr{C}^{k+2,\alpha}(V, \mathfrak{X})\| \leq c\|f, \mathscr{C}^{k,\alpha}(V, \mathfrak{X})\| + c'$$

where the constant c' *is independent of* f *and depends on a bound for the solution of the homogeneous equation* $\mathscr{L}_{\mathfrak{X}} v = 0$.

10.1 General second order linear subelliptic equation

Consider the linear second order equation of the form

$$\mathscr{P}_{\mathfrak{X}}(u) = \sum_{i,j=1}^{m} a_{ij}(x) X_i X_j u + \sum_{j=1}^{m} a_j(x) X_j u + a_0(x) u = f$$

where $a_{ij},\ a_j,\ a_0 \in C^\infty(M, \mathbb{R})$ and $(a_{ij}(x))$ is a symmetric $m \times m$-matrix for all $x \in \Omega$.

Theorem 11 *Suppose the coefficients* $a_{ij},\ a_j,\ a_0 \in \mathscr{C}^{k,\alpha}(M, \mathfrak{X})$ *for* $k \in \mathbb{N}$ *and* $\alpha \in (0, 1)$ *such that, for all* $x \in \Omega$, $(a_{ij}(x))$ *is a positive definite* $m \times m$*-matrix. If* $u \in \mathscr{C}^{2,\alpha}(M, \mathfrak{X})$ *is a solution of* $\mathscr{P}_{\mathfrak{X}}(u) = f$ *in* M *then* $u \in \mathscr{C}^{k+2,\alpha}(M, \mathfrak{X})$ *whenever* $f \in \mathscr{C}^{k,\alpha}(M, \mathfrak{X})$

Proof For every $x^0 \in M$ there is an $R_0 > 0$ such that $B_\rho(x^0, 2R) \subset V_{x^0}$, the exponential mapping neighbourhood used to define the quasi-metric $\rho(x, y)$. Freezing the coefficients at x^0 let

$$\mathscr{P}_0 = \sum_{i.j=1}^{m} a_{ij}(x^0) X_i X_j + c_0, \quad \text{with} \quad c_0 < 0.$$

Here $A = (a_{ij}(x^0))$ is positive definite. Let $\{\lambda_1, \ldots, \lambda_m\}$ be the eigen-values of A. Then there exists a nonsingular $m \times m$-matrix Q such that

$$Q^{-1}AQ = D = \mathrm{diag}[\lambda_1^{-\frac{1}{2}}, \ldots, \lambda_m^{-\frac{1}{2}}] = (\lambda_j^{-\frac{1}{2}}).$$

Let $\mathfrak{Y} = Q\mathfrak{X}$. The operator $\mathscr{P}_0$ is transformed to the Hörmander type operator $\mathscr{L}_{\mathfrak{Y}} = \sum_{j=1}^{m} Y_j^2$ and the system of vector fields $\mathfrak{Y} = \{Y_1, \ldots, Y_m\}$ satisfy the assumptions (A1) and (A2) and we get a new quasi-metric σ (which is equivalent to ρ) defined by the geometry associated to the new system $\mathfrak{Y}$.

We also have

$$\mathscr{C}^{k,\alpha}(V, \mathfrak{X}) = \mathscr{C}^{k,\alpha}(V, \mathfrak{Y}) \text{ for all } k \in \mathbb{N} \text{ and } \alpha \in (0, 1).$$

then we have the regularity of v of class $\mathscr{C}^{k+2,\alpha}(V, \mathfrak{Y})$ for solutions of the equation $\mathscr{P}_0 v = g \in \mathscr{C}^{k,\alpha}(V, \mathfrak{Y})$ together with a precise estimate.

Finally we write

$$\mathscr{P}_{\mathfrak{X}} u = \mathscr{P}_0 u + b_0 u$$
$$= f - \sum_{ij}[a_{ij}(x) - a_{ij}(x^0)]X_i X_j u - \sum_j b_j(x) X_j u + (b_0(x) - c_0(x))u.$$

The conclusion follows using the previous result together with the estimate and the interpolation lemma. □

10.2 The general quasilinear equation

Theorem 12 *Let $u \in \mathscr{C}^{2,\alpha}(V, \mathfrak{X})$ be a solution of the quasilinear subelliptic equation*

$$\sum_{i,j=1}^{m} a_{ij}(x, u, \mathfrak{X}u) X_i X_j u + b(x, u, Xu) = 0 \ \ in \ \ M$$

where $a_{ij}, b \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^m; \mathbb{R})$ and for all $(x, u, \xi) \in \Omega \times \mathbb{R}^{m+1}$ the matrix $(a_{ij}(x, u, \xi))$ is positive definite. Then $u \in C^\infty(M)$.

Proof We argue as before in the exponential mapping neighbourhood $V = V_{x^0}$ of a point $x^0 \in M$. We take $a_{ij}(x) = a_{ij}(x, u(x), Xu(x))$ and $f(x) = b(x, u, Xu(x))$ so that $a_{ij},\ f \in \mathscr{C}^{1,\alpha}(V; \mathfrak{X})$. By the above argument for the linear equations we get $u \in \mathscr{C}^{3,\alpha}(V, \mathfrak{X})$. This in turn implies that $a_{ij},\ f \in \mathscr{C}^{2,\alpha}(V; \mathfrak{X})$ from which we deduce from the linear equations argument that $u \in \mathscr{C}^{4,\alpha}(V; \mathfrak{X})$. The bootstrap argument can be continued to conclude that $u \in \mathscr{C}^{k,\alpha}(V; \mathfrak{X})$ for all $k \in \mathbb{N}$. Then by the embedding theorem for $\mathscr{C}^{k,\alpha}(V; \mathfrak{X})$ in the classical Hölder spaces it follows that $u \in C^\infty(V)$. □

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